REVERSE MODELLING OF STREAMFLOW IN A RECTANGULAR CHANNEL USING MUSKINGUM METHOD : A THEORETICAL APPROACH

Issa LEYE[1], Soussou SAMBOU[1], Seïdou KANE[1], Moussé Landing SANE[1], Séni TAMBA[2], Didier Maria NDIONE[1]*

[1]Université Cheikh Anta DIOP, Faculté des Sciences et Techniques, Département de Physique, Laboratoire d'hydraulique et de Mécanique des Fluides, BP 5005 Dakar, Sénégal.
[2]Université Cheikh Anta DIOP, Ecole Supérieure Polytechnique centre de Thiès.
*issa.leye@ucad.edu.sn

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ABSTRACT

Stream flow at a particular location of a river network is the response of the basin to time and space integrated runoff. Runoff – stream flow process is parameterized by rooting models. Rooting models can then be used to predict stream flow at any gauging location of the river network. Inverting stream flow to runoff process is referred as "inverse routing". Inverse routing is very important in water resources planning: control of water for irrigation, disaster prevention related to water, mapping of flood risk etc. It allows deriving time and spatially distributed runoff from stream flow at a given location. In this paper, the problem we tackle is how to derive upstream hydrographs knowing a downstream hydrograph at gauging location in the river network section. Inverse Muskingum method is applied. Inverse Muskingum parameters K and X are estimated by Lagrange multipliers. This leads to a system of two non-linear equations. This system is solved numerically using an iterative method based on the minimization of an objective function that is here the sum of squared errors. The method has been validated by comparison with the results found in literature dealing. It can be applied to the management of water levels in irrigation networks.

Keywords: inverse rooting; Muskingum method; Lagrange multipliers

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1 INTRODUCTION

Stream flow at a given gauge location in a river network derives from time and spatially distributed runoff over a river basin (Pan M., and Wood E. F., 2013 ; Carter R.W., and Godfrey R.G., 1960). Flood is defined as condition in which stream flow unexpectedly increase so as to cause financial and fatal damages. It is a natural phenomenon that human society has accepted as inevitable event. The movement of a flood wave in a river catchment is a highly complicated phenomenon of unsteady and non uniform flow (Peters J. C., 1983). Flood routing is the technique of determining the flood hydrograph at a section of a river by utilizing the data flow at one to or more upstream sections. It allows to answer the question "what is the outflow at the lower end, given the hydrograph at the upper end in a stream river". Flood routing is an important part of flood management. It is applied in the following situations: determine effects of proposed flood – control projects, forecast river stages, schedule operations in a hydro-electric power system according to the predicted progress of the flood, risk evaluation, impact of landuse change. It is also important in the design of flood protection measures such as spillway or reservoir, to estimate how the proposed measures will affect the behavior of flood waves in river, so that adequate and economic protection solutions may be found (Mazi K., and Kousis A.D., 2010). Flood routing uses a mathematical method for predicting the changing magnitude, shape, and celerity of a flood wave which propagates through a river (Doiphode S. L., and Oak R. A., 2012). Flood routing methods may be classified as either hydrologic or hydraulic. Hydrologic routing use principle of continuity and relation between discharge and temporary storage of excess volume during the flood period. Hydrological issues concern prevention and control of floods and have been discussed for years in the world which indicates their importance (Fasahat V., et al., 2013). Among the hydrologic flood routing methods, the Muskingum method has been extensively applied in river engineering practices (Mesfin. H. T., 2008). Hydraulic methods of routing involve numerical solutions of the convective diffusion models or one dimensional Saint-Venant of gradually varied unsteady flow in open channels.

Discharge data are not always available at sites where they are required for sustainable use of water resources and for design flood. Reverse routing process is a procedure of determining an upstream flow conditions based upon the knowledge of downstream flow conditions and the hydraulics characteristics of a river reach (Zucco G., et al, 2015). The following applications fit to reverse routing (Das A., 2009) : tracing back the flood devastations using available floods hydrographs at downstream location, recalibrating local flow characteristics altered by human activities at a given reach of a river using unaffected downstream gauge, verify upstream gauging site using the discharge at immediate downstream gauging site, recalibrate upstream gauge when a new gauge is installed between two existing gauge, verifying reservoir releases, reproducing hydrographs. In flood control, reverse flood routing is also used to plan upstream hydrograph flow from known downstream flow hydrograph.

In this paper, we propose an analytical method of estimating Muskingum parameters K and X in reverse flood routing in a rectangular, based on minimization of objective function with constraints. Sum of square of errors is the objective function, whereas constraints are verification of two formulations of linear storage equation. Objective function and constraints are related using Lagrange transformation and Lagrange multipliers. Minimizing Lagrange transformation lead to a system of two non linear equations relating K and X. This system is solved iteratively. Resutls we obtain are compared with those in the literature.

2 LITERATURE REVIEW

Many approaches have been used in river or channel inverse flood routing: Saint – Venant equations (Mustafa H. A., et al, 2014 ; Szymkiewicz R., 2008); diffusive wave equation (Koussis A. D., et al, 2012); Hayamiequation (Dooge J., and Bruen M., 2005); Muskingum method (Das A, 2009) and Muskingun Cunge method (Mustafa H. A., and Kadhim N. K., 2013). The Muskingum method is very popular in reverse flood routing.

The Muskingum equation is based on two equations :

- the continuity equation which expresses the conservation of volume, is written under the following form (equation 1.1)

$$\frac{dV_{s(t)}}{dt} = Q_{i(t)} - Q_{o(t)} \qquad (1.1)$$

- the storage equation expressed in a form of a linear relation between the variation of storage and the weighted average of the input and output (equation 1.2).

$$V_{s(t)} = K\left[XQ_{i(t)} + (1 - X)Q_{o(t)}\right] \qquad (1.2)$$

in which, s = the channel storage; i = the inflow; o = the outflow; t = the time

$V_{s(t)}$: storage volume (m^3)
$Q_{i(t)}$: inflow at time t (m^3/s)
$Q_{o(t)}$: outflow at time t (m^3/s)
K : proportionality coefficient (s)
X : dimensionless weighting factor

A representation of the linear storage is given in figure1 :

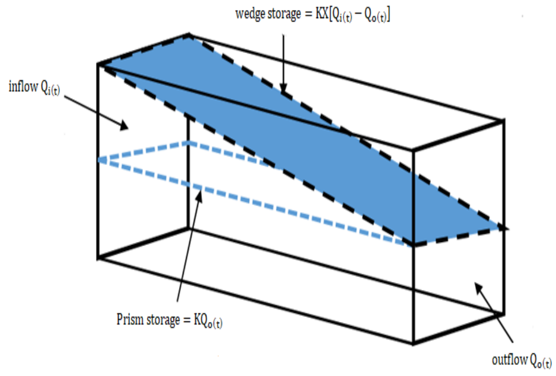


Figure 1. Principle of the Muskingum equation storage model (Lèye I., 2015).

Equation (1.1) and equation (1.2) are discretized using the finite difference method (equations 1.3 and 1.4) and the trapezoidal method (equation 1.4, Figure 2) for any function f(x,t) of space and time.

$$\frac{df}{dt} = \frac{f^{n+\Delta t} - f^{n}}{\Delta t} \quad (1.3)$$

$$f = \frac{f^{n+\Delta t} + f^{n}}{2} \quad (1.4)$$

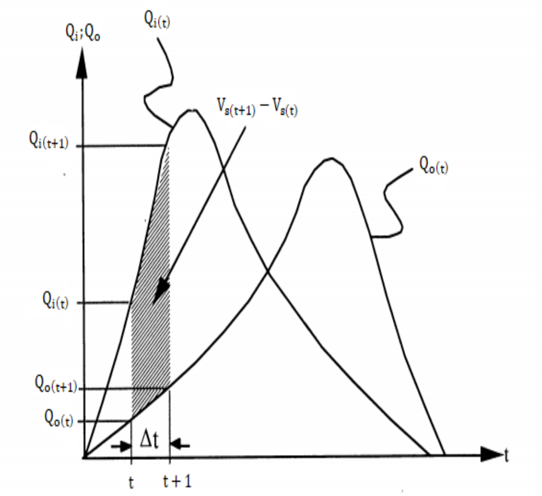


Figure 2. Evaluation of the variation of water storage in a reach (Muzy, A., and Higy, C., 1998)

The integration of the continuity equation (1.1) between times t and t + Δt using equation (1.3) for the left member and (1.4) for the right member and then using equation (1.3) for the left and right members of the linear storage equation (1.2) which leads to equation (1.5) and (1.6), where Δt is the time interval.

$$V_{s(t+1)} - V_{s(t)} = \left(\frac{Q_{i(t)} + Q_{i(t+1)}}{2}\right)\Delta t - \left(\frac{Q_{o(t)} + Q_{o(t+1)}}{2}\right)\Delta t \quad (1.5)$$

$$V_{s(t+1)} - V_{s(t)} = K\left[X\left(Q_{i(t+1)} - Q_{i(t)}\right) + (1-X)\left(Q_{o(t+1)} - Q_{o(t)}\right)\right] \quad (1.6)$$

The equalization of equations 1.5 and 1.6 leads to equation 1.7 after arrangement.

$$Q_{i(t)} = \frac{\left(KX - \frac{\Delta t}{2}\right)}{\left(KX + \frac{\Delta t}{2}\right)}Q_{i(t+1)} + \frac{\left(-K + KX + \frac{\Delta t}{2}\right)}{\left(KX + \frac{\Delta t}{2}\right)}Q_{o(t)} + \frac{\left(K - KX + \frac{\Delta t}{2}\right)}{\left(KX + \frac{\Delta t}{2}\right)}Q_{o(t+1)} \quad (1.7)$$

Let us set

$$C_{01} = \frac{(KX - 0{,}5\Delta t)}{(KX + 0{,}5\Delta t)} \quad (1.8)$$

$$C_{11} = \frac{(-K + KX + 0{,}5\Delta t)}{(KX + 0{,}5\Delta t)} \quad (1.9)$$

$$C_{21} = \frac{(K - KX + 0{,}5\Delta t)}{(KX + 0{,}5\Delta t)} \quad (1.10)$$

Then the following equation 1.11 allows to calculate the input flow

$$Q_{i(t)} = C_{01}Q_{i(t+1)} + C_{11}Q_{o(t)} + C_{21}Q_{o(t+1)} \quad (1.11)$$

where

$$C_{01} + C_{11} + C_{21} = 1 \quad (1.12)$$

3 THE PARAMETER (K AND X) ESTIMATION

A statistical method based on the minimization of an objective function is generally used. In what follows, the objective function is the sum of the squares of errors (SSE) calculated as differences between the observed inflows $\bar{Q}_{i(t)}$ and the calculated inflows $Q_{i(t)}$:

$$SSE = \sum_{t=1}^{n+1}\left(Q_{i(t)} - \bar{Q}_{i(t)}\right)^2 \quad (1.13)$$

where n is the total number of observations. The parameters K and X are calculated so as they verify equations (1.5) and (1.6). We face a problem of minimization of an objective function (1.13) with constraints. We have here only one constraint, that is obtained by setting to zero the sum of differences of equation (1.5) and (1.6). We obtain by this way (1.14) . These constrained optimization problem is transformed into unconstrained problem by using Lagrange multipliers (equation 1.15) according to Das (2009).

$$\sum_{t=1}^{n}\Big[K\big[XQ_{i(t+1)}+(1-X)Q_{o(t+1)}\big]-K\big[XQ_{i(t)}+(1-X)Q_{o(t)}\big] - \frac{\Delta t}{2}\{[Q_{i(t+1)}+Q_{i(t)}]-[Q_{o(t+1)}+Q_{o(t)}]\}\Big]=0 \qquad (1.14)$$

$$L=\sum_{t=1}^{n+1}(Q_{i(t)}-\bar{Q}_{i(t)})^2 + \sum_{t=1}^{n}\Big[K\big[XQ_{i(t+1)}+(1-X)Q_{o(t+1)}\big]-K\big[XQ_{i(t)}+(1-X)Q_{o(t)}\big] - \frac{\Delta t}{2}\{[Q_{i(t+1)}+Q_{i(t)}]-[Q_{o(t+1)}+Q_{o(t)}]\}\Big]\lambda_t=0 \qquad (1.15)$$

where L is the transformed lagrangian objective fonction and λ_t = the Lagrange multipliers corresponding to the constraints of order t.

The cancellation of the derivatives of the function L with respect to $\mathbf{Q_{i(t)}}$, K and X (equations 1.16, 1.18, 1.20) gives the equations below (1.17, 1.19, 1.21).

$$\frac{\partial L}{\partial Q_{i(t)}}=0 \qquad (1.16)$$

$$\lambda_t=\sum_{l=1}^{t}\frac{2(-1)^{l-1}\left[Q_{i(t-(l-1))}-\bar{Q}_{i(t-(l-1))}\right]\left(KX-\frac{\Delta t}{2}\right)^{l-1}}{\left(-KX-\frac{\Delta t}{2}\right)^{l}} \qquad (1.17)$$

for l = 1, 2, 3……, t.

$$\frac{\partial L}{\partial K}=0 \qquad (1.18)$$

$$\sum_{t=1}^{n}\{[XQ_{i(t+1)}+(1-X)Q_{o(t+1)}]-[XQ_{i(t)}+(1-X)Q_{o(t)}]\}\lambda_t=0 \qquad (1.19)$$

$$\frac{\partial L}{\partial X}=0 \qquad (1.20)$$

$$\sum_{t=1}^{n}K\{[Q_{i(t+1)}-Q_{o(t+1)}]-[Q_{i(t)}-Q_{o(t)}]\}\lambda_t=0 \qquad (1.21)$$

The replacement of λ_t (equation 1.17) by its expression in equations (1.19) and (1.21) leads to the equation (1.22) and (1.23).

$$\sum_{t=1}^{n}\{[XQ_{i(t+1)}+(1-X)Q_{o(t+1)}]-[XQ_{i(t)}+(1-X)Q_{o(t)}]\} \times\sum_{l=1}^{t}\frac{2(-1)^{l-1}\left[Q_{i(t-(l-1))}-\bar{Q}_{i(t-(l-1))}\right]\left(KX-\frac{\Delta t}{2}\right)^{l-1}}{\left(-KX-\frac{\Delta t}{2}\right)^{l}} =0 \qquad (1.22)$$

$$\sum_{t=1}^{n}K\{[Q_{i(t+1)}-Q_{o(t+1)}]-[Q_{i(t)}-Q_{o(t)}]\} \times\sum_{l=1}^{t}\frac{2(-1)^{l-1}\left[Q_{i(t-(l-1))}-\bar{Q}_{i(t-(l-1))}\right]\left(KX-\frac{\Delta t}{2}\right)^{l-1}}{\left(-KX-\frac{\Delta t}{2}\right)^{l}} =0 \qquad (1.23)$$

Thus, a nonlinear system of two equations (1.22, 1.23) with two unknowns K and X is obtained, which we solve by an iterative method. The details of the derivations are given in annex A, and the iterative procedure is presented below.

- 1. given ΔX and ΔK
- 2. set X_0
- 3. set K_0
- 4. Introduced X_0 and K_0 into equation (1.22) to start the iterations
- 5. If equation (1.22) is satisfied, go to step 8
- 6. If equation (1.22) is not satisfied, we go to step 7
- 7. X_0 is replaced by $X_0 + \Delta X_0$, return to step 4
- 8. If equation (1.23) is satisfied, go to step 9
- 9. Calculate the coefficients of Muskingum C_{01}, C_{11} and C_{21}

- 10. Upstream hydrograph is calculated $Q_{i(t)}$

4 RESULTS AND DISCUSSIONS

The hydraulic system proposed by Wilson (1974) is used. It consists of a single input and a single output channel. The values of input and output Wilson's observed hydrographs are indicated in Table 1, columns 2 and 3. The corresponding input and output plots are presented in figure 3.

Table 1. observed outflow and and calculated hydrographs.

	Observed outflow and inflow hydrographs (Wilson, 1974) (m^3/s)		Calculated inflow hydrographs (m^3/s)			
Time (hr)	$\bar{Q}_{o(t)}$	$\bar{Q}_{i(t)}$	**nosliW 1974**	**Das 2004**	**Das 2009**	**Lèye et al**
0	22	22	30	20	24	23
6	21	23	42	27	31	33
12	21	35	63	45	45	66
18	26	71	80	61	59	95
24	34	103	93	76	71	103
30	44	111	103	88	82	102
36	55	109	107	97	89	96
42	66	100	103	99	92	84
48	75	86	95	98	90	72
54	82	71	81	89	84	62
60	85	59	64	77	74	51
66	84	47	48	64	63	44
72	80	39	32	49	51	37
78	73	32	19	36	39	33
84	64	28	9	24	30	28
90	54	24	5	16	22	25
96	44	22	6	14	19	23
102	36	21	7	13	17	21
108	30	20	8	12	16	20
114	25	19	11	14	16	20
120	22	19	13	13	15	18
126	19	18	18	18	18	18

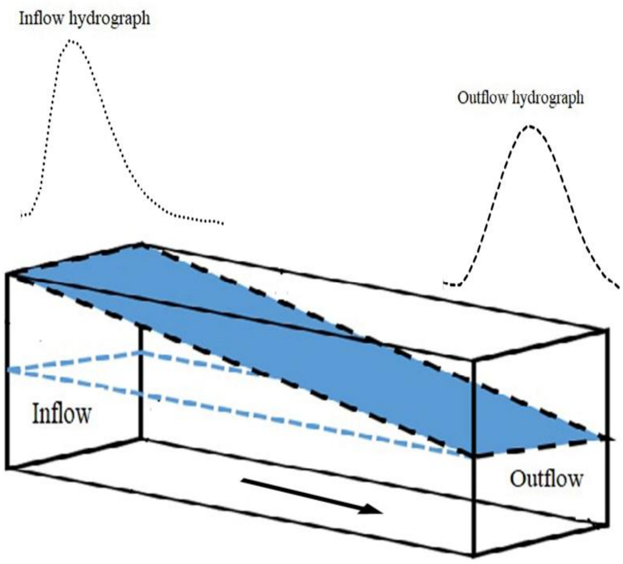


Figure 3 . Rectangular channel with inflow and outflow observed hydrographs (Wilson, 1974)

The problem of inverse model is to calculate the inflow hydrograpgh corresponding to the outflow hydrograph. The values of inflow and outflow observed Wilson's hydrograpgh are introduced in equations 1.22 and 1.23 to calculate Muskingum coefficients K and X according to the iterative procedure 1 to 8. Then equations 1.8, 1.9 and 1.10 are used to estimate coefficients C_{01}, C_{11} and C_{21}. These last coefficients allow to determine upstream input hydrograph $Q_{i(t)}$ according to equation 1.11. We present in Table 1, the values of the input hydrograph calculated by our method (column 7). For comparison, we add the inflow hydrograph calculated by Wilson (1974) (column 4), and those calculated by Das (2004, 2009) (columns 5 and 6) in table 1.

Reverse Muskingum parameters K and X obtained by Das (2004, 2009) and Wilson (1974) and using the method we propose are indicated in Table 2. The values of K and X we found are in the order of magnitude of the values found by Wilson (1974) and Das (2004, 2009)

Table 2. Values of the parameters K and X

Paramet ers	Wilso n	Das 2004	Das 2009	Lèye et al
K (hours)	36.00 00	24.38 04	22.00 11	25.00 00
X	0.250 0	0.201 0	0.339 6	0.370 0

The accuracy of the different models has been verified using the Mean Percentage Relative Error (MPRE). It is a measure to the difference between predicted and observed values. The MPRE is defined as:

$$MPRE = \frac{\sum\left[\left(\frac{\text{calculated inflow} - \text{observed inflow}}{\text{observed inflow}}\right) x100\right]}{\text{number of results}} \qquad (1.24)$$

Values of MPRE are presented in table 3. According to this criterion, the method we present suit very well since it is better than that of Wilson (1974) and Das (2014)

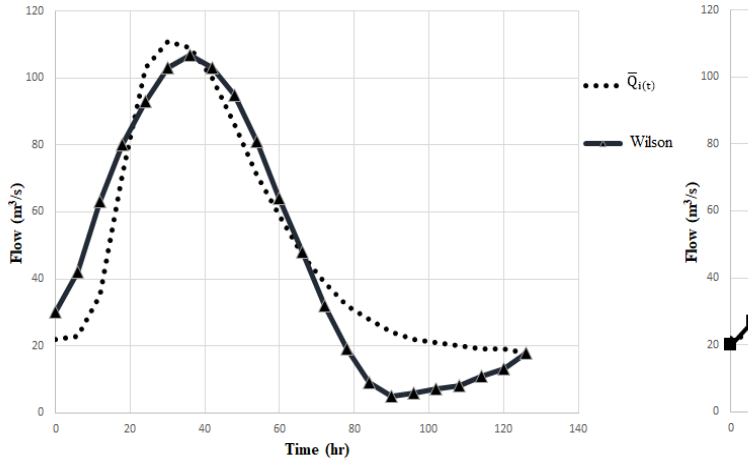
Table 3. Accuracies comparison of the models.

Criterion of accuracy	Wilson	Das 2004	Das 2009	Lèye et al
MPRE (%)	-11.3	- 5.1	0. 7	4. 2

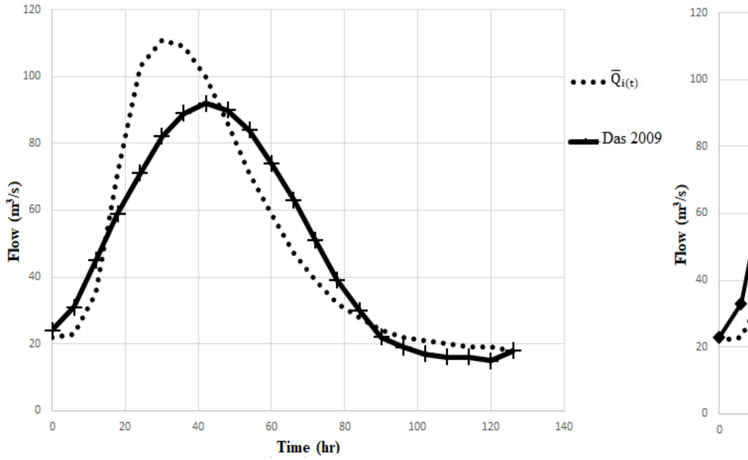
The plots of the observed inflow hydrograph (Wilson, 1974) and the inflow hydrographs caluclated by different methods are compared in figure 4a (Wilson, 1974), figure 4b (Das, 2004), figure 4c (Das, 2009), figure 4d (Lèye et al, 2018).

By comparing the inflow hydrograph calculated with the iterative procedure presented in this paper to the Wilson's observed inflow hydrograph, it appears that:

- peak flow is underestimated
- time to peak is the lowest among all methods
- recession curve is well restituted, particularly in the lower part of the calculated hydrograpgh



a : Plots of observed and Wilson's 1974 hydrographs b : Plots of observed and Das 2004 hydrographs



c : Plots of observed and Das 2009 hydrographs d : Plots of observed and Lèye et al hydrographs

Figure 4 : Plots of observed and calculated Inflow hydrographs by Wilson (a); Das , 2004 (b); Das 2009 (c), Lèye et al (d)

5 CONCLUSIONS

The objective of this work was to use the inverse modeling to calculate the hydrograph upstream knowing hydrograph downstream in a section of a rectangular channel. The Reverse Muskngum method have been applied. A theoretical approach, based on least squared methods is developed. The objective function to minimize is a combination of sum of square of errors and linear storage equation with lagrange multipliers. This leads to a system of two non linear equations relationg the unknown K and X.. Iterative method have been developed to solve this system. The results obtained using the same data as Das (2009), even if they show some differences, can be considered as satisfactory. We intend to improve the iterative procedure used and to use the method thus developed to study the problem of satisfying water requirements in irrigation systems.

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NOTATION

The following symbols are used in this paper :

f : given function of x and t

i : the inflow

K : proportionality coefficient (s)

L : the transformed lagrangian objective fonction

MPRE: Mean Percentage Relative Error

n : the total number of time stages in the stream flow routing process

o : the outflow

$Q_{i(t)}$: calculated inflow at time t (m^3/s)

$\bar{Q}_{i(t)}$: observed inflow at time t (m3/s)

$Q_{i(t+1)}$: inflow at time t +1 (m^3/s)

$Q_{o(t)}$: outflow at time t (m^3/s)

$Q_{o(t+1)}$: outflow at time t + 1 (m^{3}s)

s : the channel storage

SSE : sun of square of error

t : the time

Δt : time interval

$V_{s(t)}$: storage volume (m^3)

X : dimensionless weighting factor

λ_t : Lagrange multiplier

APPENDIX A

Analytical value of the lagrange multipliers

A recurrence method is applied to equation A.1. The method we propose consists in establishing the analytic expression (A.1) of L at the rank n + 1 and we obtain the expression (A.2) :

$$L = \sum_{t=1}^{n+1}\left(Q_{i(t)} - \bar{Q}_{i(t)}\right)^2 + \sum_{t=1}^{n}\left[K\left[XQ_{i(t+1)} + (1-X)Q_{o(t+1)}\right] - K\left[XQ_{i(t)} + (1-X)Q_{o(t)}\right] - \frac{\Delta t}{2}\left\{\left[Q_{i(t+1)} + Q_{i(t)}\right] - \left[Q_{o(t+1)} + Q_{o(t)}\right]\right\}\right]\lambda_t = 0 \quad (A.1)$$

We develop equation A.1 for t=1, …, n+1 (equation A2)

$$\begin{aligned} L = {} & \left[Q_{i(1)} - \bar{Q}_{i(1)}\right]^2 + \left[Q_{i(2)} - \bar{Q}_{i(2)}\right]^2 + \left[Q_{i(3)} - \bar{Q}_{i(3)}\right]^2 + \cdots + \left[Q_{i(n)} - \bar{Q}_{i(n)}\right]^2 \\ & + \left[Q_{i(n+1)} - \bar{Q}_{i(n+1)}\right]^2 \\ & + \left[K\left[XQ_{i(2)} + (1-X)Q_{o(2)}\right] - K\left[XQ_{i(1)} + (1-X)Q_{o(1)}\right] \right.\\ & \left. - \frac{\Delta t}{2}\left\{\left[Q_{i(2)} + Q_{i(1)}\right] - \left[Q_{o(2)} + Q_{o(1)}\right]\right\}\right]\lambda_1 \\ & + \left[K\left[XQ_{i(3)} + (1-X)Q_{o(3)}\right] - K\left[XQ_{i(2)} + (1-X)Q_{o(2)}\right] \right.\\ & \left. - \frac{\Delta t}{2}\left\{\left[Q_{i(3)} + Q_{i(2)}\right] - \left[Q_{o(3)} + Q_{o(2)}\right]\right\}\right]\lambda_2 \\ & + \left[K\left[XQ_{i(4)} + (1-X)Q_{o(4)}\right] - K\left[XQ_{i(3)} + (1-X)Q_{o(3)}\right] \right.\\ & \left. - \frac{\Delta t}{2}\left\{\left[Q_{i(4)} + Q_{i(3)}\right] - \left[Q_{o(4)} + Q_{o(3)}\right]\right\}\right]\lambda_3 + \cdots \\ & + \left[K\left[XQ_{i(n+1)} + (1-X)Q_{o(n+1)}\right] - K\left[XQ_{i(n)} + (1-X)Q_{o(n)}\right] \right.\\ & \left. - \frac{\Delta t}{2}\left\{\left[Q_{i(n+1)} + Q_{i(n)}\right] - \left[Q_{o(n+1)} + Q_{o(n)}\right]\right\}\right]\lambda_n \end{aligned} \quad (A.2)$$

The values of λ_t are determined from the following condition :

$$\frac{\partial L}{\partial Q_{i(t)}} = 0 \quad (A.3)$$

We first set t to 1, 2, 3, …, n and use equation A.3. We obtain equations A.1, A.2., A.3, …, A7.

$$\frac{\partial L}{\partial Q_{i(1)}} = -2\left[Q_{i(1)} - \bar{Q}_{i(1)}\right] + \left(-KX - \frac{\Delta t}{2}\right)\lambda_1 = 0 \quad (A.4)$$

$$\frac{\partial L}{\partial Q_{i(2)}} = -2\left[Q_{i(2)} - \bar{Q}_{i(2)}\right] + \left(KX - \frac{\Delta t}{2}\right)\lambda_1 + \left(-KX - \frac{\Delta t}{2}\right)\lambda_2 = 0 \quad (A.5)$$

$$\frac{\partial L}{\partial Q_{i(3)}} = -2\left[Q_{i(3)} - \bar{Q}_{i(3)}\right] + \left(KX - \frac{\Delta t}{2}\right)\lambda_2 + \left(-KX - \frac{\Delta t}{2}\right)\lambda_3 = 0 \quad (A.6)$$

$$\frac{\partial L}{\partial Q_{i(n)}} = -2\left[Q_{i(n)} - \bar{Q}_{i(n)}\right] + \left(KX - \frac{\Delta t}{2}\right)\lambda_{n-1} + \left(-KX - \frac{\Delta t}{2}\right)\lambda_n = 0 \quad (A.7)$$

For each step we derive the expression of Lagrange multiplier A8, A9, A10, A11 and finally the general form A12.

$$\lambda_1 = \frac{2\left[Q_{i(1)} - \bar{Q}_{i(1)}\right]}{\left(-KX - \frac{\Delta t}{2}\right)} \quad (A.8)$$

$$\lambda_2 = \frac{2\left[Q_{i(2)} - \bar{Q}_{i(2)}\right]}{\left(-KX - \frac{\Delta t}{2}\right)} - \frac{2\left[Q_{i(1)} - \bar{Q}_{i(1)}\right]\left(KX - \frac{\Delta t}{2}\right)}{\left(-KX - \frac{\Delta t}{2}\right)^2} \qquad (A.9)$$

$$\lambda_3 = \frac{2\left[Q_{i(3)} - \bar{Q}_{i(3)}\right]}{\left(-KX - \frac{\Delta t}{2}\right)} - \frac{2\left[Q_{i(2)} - \bar{Q}_{i(2)}\right]\left(KX - \frac{\Delta t}{2}\right)}{\left(-KX - \frac{\Delta t}{2}\right)^2} + \frac{2\left[Q_{i(1)} - \bar{Q}_{i(1)}\right]\left(KX - \frac{\Delta t}{2}\right)^2}{\left(-KX - \frac{\Delta t}{2}\right)^3} \quad (A.10)$$

$$\lambda_n = \frac{2\left[Q_{i(n)} - \bar{Q}_{i(n)}\right]}{\left(-KX - \frac{\Delta t}{2}\right)} - \frac{2\left[Q_{i(n-1)} - \bar{Q}_{i(n-1)}\right]\left(KX - \frac{\Delta t}{2}\right)}{\left(-KX - \frac{\Delta t}{2}\right)^2} + \frac{2\left[Q_{i(n-2)} - \bar{Q}_{i(n-2)}\right]\left(KX - \frac{\Delta t}{2}\right)^2}{\left(-KX - \frac{\Delta t}{2}\right)^3} + \cdots + \frac{2\left[Q_{i(2)} - \bar{Q}_{i(2)}\right]\left(KX - \frac{\Delta t}{2}\right)^{n-2}}{\left(-KX - \frac{\Delta t}{2}\right)^{n-1}} + \frac{2\left[Q_{i(1)} - \bar{Q}_{i(1)}\right]\left(KX - \frac{\Delta t}{2}\right)^{n-1}}{\left(-KX - \frac{\Delta t}{2}\right)^{n}} \quad (A.11)$$

So we can write the expression of λ_t in the form :

$$\lambda_t = \sum_{l=1}^{t} \frac{2(-1)^{l-1}\left[Q_{i(t-(l-1))} - \bar{Q}_{i(t-(l-1))}\right]\left(KX - \frac{\Delta t}{2}\right)^{l-1}}{\left(-KX - \frac{\Delta t}{2}\right)^{l}} \qquad (A.12)$$

With l = 1,2 ....,t and t = 1,2 ...., n

APPENDIX B

- Equation leading to parameter K.

We derivate the Lagrange transformation function.

The first term is equal to zero.
The last gives after simplifications.

$$\frac{\partial L}{\partial K} = 0 \qquad (B.1)$$

$$\frac{\partial L}{\partial K} = \sum_{t=1}^{n+1} \frac{\partial}{\partial K}\left[Q_{i(t)} - \bar{Q}_{i(t)}\right]^2 + \sum_{t=1}^{n} \frac{\partial}{\partial K}\Big[K\left[XQ_{i(t+1)} + (1-X)Q_{o(t+1)}\right] - K\left[XQ_{i(t)} + (1-X)Q_{o(t)}\right] - \frac{\Delta t}{2}\left\{\left[Q_{i(t)} + Q_{i(t+1)}\right] - \left[Q_{o(t)} + Q_{o(t+1)}\right]\right\}\Big]\lambda_t \qquad (B.2)$$

$$\sum_{t=1}^{n+1} \frac{\partial}{\partial K}\left[Q_{i(t)} - \bar{Q}_{i(t)}\right]^2 + \sum_{t=1}^{n}\Big[\frac{\partial}{\partial K}K\left[XQ_{i(t+1)} + (1-X)Q_{o(t+1)}\right] - \frac{\partial}{\partial K}K\left[XQ_{i(t)} + (1-X)Q_{o(t)}\right] - \frac{\Delta t}{2}\left\{\frac{\partial}{\partial K}\left[Q_{i(t)} + Q_{i(t+1)}\right] - \frac{\partial}{\partial K}\left[Q_{o(t)} + Q_{o(t+1)}\right]\right\}\Big]\lambda_t = 0 \qquad (B.3)$$

$$\sum_{t=1}^{n}\left\{\left[XQ_{i(t+1)} + (1-X)Q_{o(t+1)}\right] - \left[XQ_{i(t)} + (1-X)Q_{o(t)}\right]\right\}\lambda_t = 0 \qquad (B.4)$$

Analytical expression of X

$$\frac{\partial L}{\partial X} = 0 \qquad (B.5)$$

$$\frac{\partial L}{\partial X} = \sum_{t=1}^{n+1} \frac{\partial}{\partial X}\left[Q_{i(t)} - \bar{Q}_{i(t)}\right]^2 + \sum_{t=1}^{n} \frac{\partial}{\partial X}\Big[K\left[XQ_{i(t+1)} + (1-X)Q_{o(t+1)}\right] - K\left[XQ_{i(t)} + (1-X)Q_{o(t)}\right] - \frac{\Delta t}{2}\left\{\left[Q_{i(t)} + Q_{i(t+1)}\right] - \left[Q_{o(t)} + Q_{o(t+1)}\right]\right\}\Big]\lambda_t \qquad (B.6)$$

$$\sum_{t=1}^{n+1} \frac{\partial}{\partial X}\left[Q_{i(t)} - \bar{Q}_{i(t)}\right]^2 + \sum_{t=1}^{n}\Big[\frac{\partial}{\partial X}K\left[XQ_{i(t+1)} + (1-X)Q_{o(t+1)}\right] - \frac{\partial}{\partial X}K\left[XQ_{i(t)} + (1-X)Q_{o(t)}\right] - \frac{\Delta t}{2}\left\{\frac{\partial}{\partial X}\left[Q_{i(t)} + Q_{i(t+1)}\right] - \frac{\partial}{\partial X}\left[Q_{o(t)} + Q_{o(t+1)}\right]\right\}\Big]\lambda_t = 0 \qquad (B.7)$$

$$\sum_{t=1}^{n} K\left\{\left[Q_{i(t+1)} - Q_{o(t+1)}\right] - \left[Q_{i(t)} - Q_{o(t)}\right]\right\}\lambda_t = 0 \qquad (B.8)$$

If we replace the expression of λ_t (appendix A, equation A.12) in equation (B.4) and (B.8) we get a system of two equations with two unknowns K and X.

$$\sum_{t=1}^{n}\left\{\left[XQ_{i(t+1)}+(1-X)Q_{o(t+1)}\right]-\left[XQ_{i(t)}+(1-X)Q_{o(t)}\right]\right\}$$

$$\text{x}\sum_{l=1}^{t}\frac{2(-1)^{l-1}\left[Q_{i\left(t-(l-1)\right)}-\bar{Q}_{i\left(t-(l-1)\right)}\right]\left(KX-\frac{\Delta t}{2}\right)^{l-1}}{\left(-KX-\frac{\Delta t}{2}\right)^{l}}$$

$$=0 \qquad (B.9)$$

$$\sum_{t=1}^{n}K\left\{\left[Q_{i(t+1)}-Q_{o(t+1)}\right]-\left[Q_{i(t)}-Q_{o(t)}\right]\right\}$$

$$\text{x}\sum_{l=1}^{t}\frac{2(-1)^{l-1}\left[Q_{i\left(t-(l-1)\right)}-\bar{Q}_{i\left(t-(l-1)\right)}\right]\left(KX-\frac{\Delta t}{2}\right)^{l-1}}{\left(-KX-\frac{\Delta t}{2}\right)^{l}}$$

$$=0 \qquad (B.10)$$